1. You are going to play 2 games of chess with an opponent whom you have never played against before (for the sake of this problem). Your opponent is equally likely to be a beginner, intermediate, or a master. Depending on   
   (a) What is your probability of winning the first game?   
   (b) Congratulations: you won the first game! Given this information, what is the probability that you will also win the second game   
   (c) Explain the distinction between assuming that the outcomes of the games are independent and assuming that they are conditionally independent given the opponent’s skill level. Which of these assumptions seems more reasonable, and why?

Solution

Let's break down the problem and solve each part step by step.

### (a) What is your probability of winning the first game?

Given that the opponent can be a beginner, intermediate, or master with equal probability, let’s assign probabilities of winning against each type of opponent:

- \*\*Probability of winning against a beginner\*\*: Let's assume this probability is \( P(\text{Win}|\text{Beginner}) = p\_B \).

- \*\*Probability of winning against an intermediate\*\*: Let's assume this probability is \( P(\text{Win}|\text{Intermediate}) = p\_I \).

- \*\*Probability of winning against a master\*\*: Let's assume this probability is \( P(\text{Win}|\text{Master}) = p\_M \).

Since the opponent is equally likely to be a beginner, intermediate, or master, the overall probability of winning the first game is:

\[

P(\text{Win}) = \frac{1}{3} \times p\_B + \frac{1}{3} \times p\_I + \frac{1}{3} \times p\_M

\]

### (b) What is the probability of winning the second game given that you won the first game?

Let’s denote the opponent's skill level by \( S \), which can be beginner, intermediate, or master.

We are interested in the conditional probability \( P(\text{Win 2}|\text{Win 1}) \). Using Bayes' theorem and the law of total probability, we can express this as:

\[

P(\text{Win 2}|\text{Win 1}) = \sum\_{S \in \{\text{B, I, M}\}} P(\text{Win 2} | S) \times P(S|\text{Win 1})

\]

To find \( P(S|\text{Win 1}) \), we use Bayes' theorem:

\[

P(S|\text{Win 1}) = \frac{P(\text{Win 1}|S) \times P(S)}{P(\text{Win 1})}

\]

Where:

- \( P(S) = \frac{1}{3} \) for each skill level (since they are equally likely).

- \( P(\text{Win 1}|S) = p\_S \), where \( p\_S \) is the probability of winning against an opponent of skill level \( S \).

Substituting these into the equation:

\[

P(S|\text{Win 1}) = \frac{p\_S \times \frac{1}{3}}{P(\text{Win 1})} = \frac{p\_S}{3P(\text{Win 1})}

\]

Thus:

\[

P(\text{Win 2}|\text{Win 1}) = \sum\_{S \in \{\text{B, I, M}\}} p\_S \times \frac{p\_S}{3P(\text{Win 1})}

\]

Simplifying:

\[

P(\text{Win 2}|\text{Win 1}) = \frac{1}{3P(\text{Win 1})} \sum\_{S \in \{\text{B, I, M}\}} p\_S^2

\]

### (c) Independence vs. Conditional Independence

- \*\*Independence\*\*: Assuming the games are independent means that the outcome of the second game does not depend on the outcome of the first game. This would mean \( P(\text{Win 2}|\text{Win 1}) = P(\text{Win 2}) \).

- \*\*Conditional Independence given the opponent’s skill level\*\*: This assumption means that once we know the opponent’s skill level, the outcomes of the games are independent. In other words, \( P(\text{Win 2}|\text{Win 1}, S) = P(\text{Win 2}|S) \). This is a more realistic assumption because if the opponent’s skill level is the same for both games, then the outcomes of the games should depend on that skill level, not on the previous outcomes.

\*\*Which assumption is more reasonable?\*\*

- The assumption of \*\*conditional independence\*\* given the opponent’s skill level seems more reasonable. In reality, if you’re playing against the same opponent in both games, their skill level will heavily influence both outcomes. The outcomes of the games are not truly independent unless you are playing against different opponents or if other factors like learning during the game or psychological factors come into play, which complicates the model further.

This concludes the solution to the problem.